Lec3: 01/25/2012

Special Relativity (Brief Review):

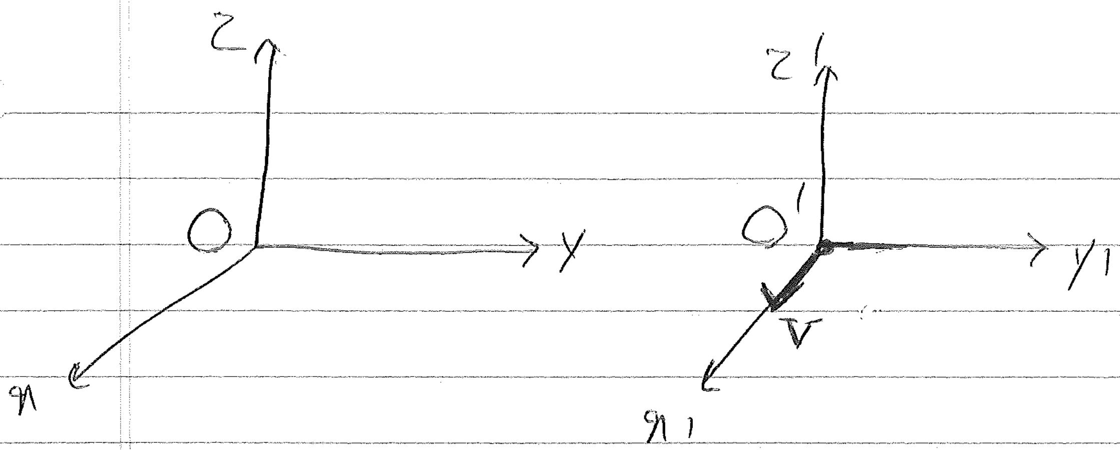
High energy astrophysics typically deals with the behavior of matter moving at close to light speed. It is therefore essential to incorporate the effects of special relativity (also general relativity, which we will discuss later). This will be important while describing the interactions between particles and radiation.

In special relativity, time and space are not separate entities as in classical physics. Instead, we have spacetime:

Classical Physics: $\underbrace{(x, y, z)}_{\text{Space}}$ & $\underbrace{t}_{\text{Time}}$

Special Relativity: $\underbrace{(x, y, z, t)}_{\text{Spacetime}}$

Transformations of spacetime coordinates between two inertial frames O, O' are given by:



$$\begin{cases} x' = \dfrac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad ct' = \dfrac{ct - \frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \\ y' = y \\ z' = z \end{cases}$$

These are called Lorentz transformation. Recall that in classical physics we have Galilean transformations:

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

It is easy to see that in the limit $\frac{v}{c} \to 0$, Lorentz transformations are reduced to Galilean transformations.

We also note that Lorentz transformations preserve the quantity ds^2 defined as follows:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The invariant ds is called the distance between two neighbouring points in the spacetime (to be precies Minkowski space). For light $ds^2 = 0$.

Introducing x^α according to:

$$x^\alpha = (ct, x, y, z) \qquad 0 \leq \alpha \leq 3 \quad ,$$

we can write:

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta = \eta^{\alpha\beta}\, dx_\alpha dx_\beta$$

$$\eta_{\alpha\beta} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \quad \eta^{\alpha\beta} \equiv (\eta_{\alpha\beta})^{-1}$$

We use Einstein Convention; having the same subscript and superscript indices in an expression implies summation.

Quantities with a subscript are obtained from those with a superscript by the help of $\eta_{\alpha\beta}$:

$$x_\alpha = \eta_{\alpha\beta}\, x^\beta$$

x^α is called a "four-vector". ds, being invariant under Lorentz transformations, is a "scalar". In general, a four-vector multiplied by a scalar yields another four-vector. Any quantity that is transformed like x^α is a four-vector:

$$x'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta}\, x^\beta = a^\alpha{}_\beta\, x^\beta$$

$$a^\alpha{}_\beta \equiv \begin{bmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_\alpha{}^\beta \equiv \left(a^\alpha{}_\beta\right)^{-1}$$

A quantity $T^{\alpha\beta}$ that is transformed according to:

$$T'^{\alpha\beta} = a^{\alpha}{}_{\gamma}\, a^{\beta}{}_{\delta}\, T^{\gamma\delta}$$

is called a rank 2 tensor. Generalization to tensors of arbitrary rank is straightforward.

Any four-vector is transformed like η^{α}. One important example is the momentum four-vector p^{α}:

$p^{\alpha} = (E, c\vec{p})$

Here E and $\vec{p}$ are defined as:

$E = \sqrt{(mc^2)^2 + c^2 |\vec{p}|^2}$, $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$

E is the energy of a particle with mass m moving with velocity $\vec{v}$, and $\vec{p}$ is its three-momentum. One can show that with these expressions, conservation of energy and momentum are valid according to all inertial frames. In the limit $\frac{v}{c} \to 0$, we restore the familiar definition of $\vec{p}$

in the classical mechanics. In this limit, the energy simply is:

$$E \simeq mc^2 + \frac{1}{2}mv^2$$

The first term is the rest energy of the particle with mass m (the celebrated $E=mc^2$ relation), and the second term is the kinetic energy of the particle.

We note that for a massless particle (like photon) we have $E = pc$ ($p \equiv |\vec{p}|$). Such a particle always moves at the speed of light (according to all inertial frames).

Another important example of a four-vector is that of the frequency and wavenumber for a photon. Using the quantum-mechanical relations $E = \hbar\omega$, $\vec{p} = \hbar \vec{k}$, we see that $(\omega, c\vec{k})$ also forms a four-vector.

This is very useful in finding the relativistic expression for the Doppler shift. Consider two inertial frames O and O', where O' is moving with velocity v in the x-direction. A photon of frequency ω according to O, hase frequency ω' according to O', where (assuming that the photon also moves in the x-direction):

$$\omega' = \frac{\omega - \frac{v}{c}(ck)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (\omega = ck)$$

Thus:

$$\omega' = \frac{\omega\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

In general, if $\vec{k}$ and $\vec{v}$ make an angle θ, we will have:

$$\omega' = \frac{\omega\left(1-\frac{v}{c}\cos\theta\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\omega(1-\beta\cos\theta)$$

Here $\beta \equiv \frac{v}{c}$ and $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (γ also called the "boost factor").

We note that if $\theta = \frac{\pi}{2}$, then $\omega' = \gamma\omega$. This is in contrast to the situation in classical physics where there is no Doppler shift for $\theta = \frac{\pi}{2}$. The difference between ω' and ω in the case of special relativity is a reflection of time dilation effect.